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2726 [1918, 303]. Proposed by E. H. MOORE, University of Chicago.

Let $\alpha_1(x, y)$, $\alpha_2(x, y)$, $\alpha_3(x, y)$, $\alpha_4(x, y)$, $\kappa_1(x, y)$, $\kappa_2(x, y)$ be six real-valued continuous functions of (x, y) over the unit-square S: $(0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1)$. Let α_1, α_2, α_3, α_4 be symmetric functions and of positive type, *i.e.*, for every real-valued continuous function $\xi(x)$ $(0 \leqq x \leqq 1)$

$$\int_0^1 \int_0^1 \xi(x)\alpha_g(x, y)\xi(y)dxdy \geqq 0.$$

Prove the inequality:

$$(J_{15}{}'J_{26}{}''J_{37}{}'''J_{48}{}^{\text{iv}} - J_{16}{}'J_{28}{}''J_{35}{}'''J_{47}{}^{\text{iv}} - J_{25}{}'J_{46}{}''J_{17}{}'''J_{38}{}^{\text{iv}} + J_{26}{}'J_{48}{}''J_{15}{}'''J_{37}{}^{\text{iv}})\kappa_1\kappa_2\kappa_1\kappa_2 \geqq 0.$$

Here on the left there are four terms of which the first is

$$\int_0^1 \cdots \int_0^1 \alpha_1(u_1u_5)\alpha_2(u_2u_6)\alpha_3(u_3u_7)\alpha_4(u_4u_8)\kappa_1(u_1u_2)\kappa_2(u_3u_4)\kappa_1(u_5u_6)\kappa_2(u_7u_8)du_1 \cdots du_8;$$

thus the eight variables of integration $u_1 \cdots u_8$ are in order the eight arguments of the four functions $\kappa_1\kappa_2\kappa_1\kappa_2$, while the "integration" symbols J indicate how the variables are to be supplied as arguments to the four functions $\alpha_1\alpha_2\alpha_3\alpha_4$, *e.g.*, $J_{26}{}''$ indicates that α_2 (the superscript $''$ determining the subscript 2) has the arguments u_2 u_6.—Indicate another inequality of this type and determine the number of such inequalities.

SOLUTION BY C. F. GUMMER, Queen's University.

It has been shown by Hilbert (see Goursat, *Cours d'analyse mathématique*, tome 3, chap. XXXII) that for every real symmetric function $\alpha_g(x, y)$ continuous in S there exist an orthogonal sequence of real functions $\phi_{g1}(x)$, $\phi_{g2}(x)$, $\cdots$, and a sequence of constants c_{g1}, c_{g2}, $\cdots$ (not zero) such that every function of the form $\int_0^1 \alpha_g(x, y)\beta(y)dy$, where $\beta(y)$ is real and continuous in $(0 \leqq y \leqq 1)$, can be represented by the series

$$\sum_i \left[c_{gi} \int_0^1 \phi_{gi}(y)\beta(y)dy \cdot \phi_{gi}(x) \right]$$

converging absolutely and uniformly in $(0 \leqq x \leqq 1)$. The function α_g is of positive type if, and only if, $c_{gi} > 0$ $(i = 1, 2, \cdots)$. The argument of Goursat, with little modification, shows that the convergence is uniform with respect to $(x, u, v, \cdots)$ in $(0 \leqq x \leqq 1, 0 \leqq u \leqq 1, \cdots)$ when for $\beta(y)$ is substituted $\gamma(y, u, v, \cdots)$ continuous in this region. This remark justifies the successive integration of the series term by term.

Applying this transformation to the first term of the inequality, we find in succession

$$\int_0^1 \alpha_1(u_1u_5)\kappa_1(u_5u_6)du_5 = \sum_i \left[c_{1i} \int_0^1 \phi_{1i}(u_5)\kappa_1(u_5u_6)du_5 \cdot \phi_{1i}(u_1) \right],$$

$$\int_0^1 \int_0^1 \alpha_1(u_1u_5)\kappa_1(u_5u_6)\alpha_2(u_2u_6)du_5du_6$$
$$= \sum_i \sum_j \left[c_{1i}c_{2j} \int_0^1 \int_0^1 \phi_{1i}(u_5)\kappa_1(u_5u_6)\phi_{2j}(u_6)du_5du_6 \cdot \phi_{1i}(u_1)\phi_{2j}(u_2) \right].$$

Since the cycle $u_1u_5u_6u_2$ is to be completed, we multiply by $\kappa_1(u_1u_2)$ and integrate with respect to u_1 and u_2, obtaining

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \alpha_1(u_1u_5)\alpha_2(u_2u_6)\kappa_1(u_1u_2)\kappa_1(u_5u_6)du_1du_2du_5du_6 = \sum_i \sum_j [c_{1i}c_{2j}\{K_1(\phi_{1i}\phi_{2j})\}^2],$$

where $K_i(\phi, \psi)$ stands for $\int_0^1 \int_0^1 \phi(u)\kappa_i(uv)\psi(v)dudv$. A similar treatment of the other four factors of the integrand shows that the first term is

$$\sum_i \sum_j \sum_k \sum_l c_{1i}c_{2j}c_{3k}c_{4l}\{K_1(\phi_{1i}\phi_{2j})K_2(\phi_{3k}\phi_{4l})\}^2.$$

The other terms of the inequality may be treated in a similar way, except that the second and third involve longer cycles, that of the second being $u_1u_6u_5u_3u_4u_7u_8u_2$. The left member is then found to be

$$\sum_i \sum_j \sum_k \sum_l c_{1i}c_{2j}c_{3k}c_{4l}\{K_1(\phi_{1i}\phi_{2j})K_2(\phi_{3k}\phi_{4l}) - K_1(\phi_{3k}\phi_{1i})K_2(\phi_{4l}\phi_{2j})\}^2,$$

which is clearly $\geqq 0$.

The number of such inequalities may be found from the coefficient of $c_{1i}c_{2j}c_{3k}c_{4l}$ in the iterated series. The first subscripts of the ϕ's in the first term of the squared factor admit 24 permutations, while the second term may be formed from it in four ways, each of the operators K_1K_2 retaining one of the ϕ's associated with it in the first term and in the opposite position. The 96 results so indicated reduce however to 48 by reason of the permutability of the terms of the squared factor. There are then 48 inequalities of precisely the same type as the one in question; but there are others of the same general type corresponding to any two permutations of the subscripts 1, 2, 3, 4, in the terms of the squared factor, making in all $\binom{24}{2}$ or 276 results, exclusive of the useless one $0 \geqq 0$ which is obtained when the same permutation is used twice. Of these, 24 are reducible inequalities of the type

$$\begin{aligned}(J_{15}'J_{26}''J_{37}'''J_{48}^{\text{iv}} - J_{15}'J_{26}''J_{38}'''J_{47}^{\text{iv}} - J_{15}'J_{26}''J_{47}'''J_{38}^{\text{iv}} + J_{15}'J_{26}''J_{48}'''J_{37}^{\text{iv}})\kappa_1\kappa_2\kappa_1\kappa_2 \\ = (J_{15}'J_{26}''\kappa_1\kappa_1) \times (J_{37}'''J_{48}^{\text{iv}} - J_{38}'''J_{47}^{\text{iv}} - J_{47}'''J_{38}^{\text{iv}} + J_{48}'''J_{37}^{\text{iv}})\kappa_2\kappa_2 \geqq 0,\end{aligned}$$

in which also each factor of the left member is separately $\geqq 0$. The other 204 are irreducible inequalities not isomorphic with the proposed formula, and falling into five classes which may be indicated by the permutation of first subscripts (*abcd*) which would be present in the second term of

$$K_1(\phi_1\phi_2)K_2(\phi_3\phi_4) - K_1(\phi_a\phi_b)K_2(\phi_c\phi_d)$$

(second subscripts being omitted) for a typical case of each class. That is, 96 correspond to the permutation (1342), 48 to (1432), 24 to (3412), 24 to (3421), and 12 to (2143).

2761 [1919, 124]. Proposed by W. W. DENTON, University of Michigan.

Find the lengths of the side of an equilateral triangle whose vertices are at given distances *a*, *b*, *c* from a given point.

I. Solution by C. E. Mange, Junior, Washington University.

To construct the triangle.

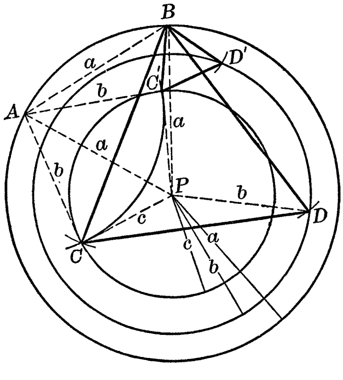
With given point, *P*, as a center, describe circles with radii *a*, *b*, and *c*, respectively. Assume $a > b > c$. Construct chord *AB* in circle (*a*) equal to *a*, and with *A* as a center describe an arc with radius *b* intersecting circle (*c*) in *C* and *C'*. With *B* as a center describe an arc of radius *BC* intersecting circle (*b*) in *D*.[1] Draw *BC*, *CD*, and *DB*. The triangle *BCD* is equilateral and fulfils the required conditions. (Similarly for the triangle *BC'D'*.)

Proof: Draw *AC*, *AP*, *AB*, *BP*, *CP*, and *DP*.

Now $AB = BP = PA$, by construction; therefore, $\triangle ABP$ is equilateral, and $\angle PBA = 60°$.

Since $\triangle ACB = \triangle BPD$, $\angle ABC = \angle PBD$; and hence $\angle ABC + \angle CBP = 60° = \angle CBD$. And since $CB = BD$, by construction, *CBD* is equilateral.

Area of



$$ACP = A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)},$$

$$\angle BAP = 60°, \qquad \text{and} \qquad \angle PAC = \text{arc sin}\,\frac{2A}{ba}.$$

In triangle *BAC*,

$$\begin{aligned} BC &= \sqrt{b^2 + a^2 - 2ba\cos\left[60° + \text{arc sin}\,\frac{2A}{ba}\right]} \\ &= \sqrt{b^2 + a^2 - 2ba\left[\frac{1}{2}\cdot\frac{b^2 + a^2 - c^2}{2ba} - \frac{\sqrt{3}}{2}\cdot\frac{2A}{ba}\right]} \\ &= \sqrt{\frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}A}. \end{aligned}$$

$$BC' = \sqrt{\frac{a^2 + b^2 + c^2}{2} - 2\sqrt{3}A}.$$

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[1] *D* is that intersection which lies on the opposite side of *BP* from *A*, so that the four lines from *B* are in the order *BA*, *BC*, *BP* and *BD*. *D'* is the intersection determined in the same way when an arc of radius *BC'* is described.—Editors.